Applied Mathematics Letters 25 (2012) 157–162

Contents lists available at SciVerse ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

The existence of a positive solution to a second-order delta–nabla p-Laplacian BVP on a time scale

Christopher S. Goodrich *

Department of Mathematics, University of Nebraska-Lincoln, Lincoln, NE 68588, USA

ARTICLE INFO

Article history:
Received 5 August 2010
Received in revised form 26 July 2011
Accepted 4 August 2011

Keywords:
Time scales
Delta–nabla dynamic equation
One-dimensional p-Laplacian
Sign-changing coefficients
Cone

ABSTRACT

In this work we consider a second-order delta–nabla dynamic boundary value problem of the form $\left(\phi_p\left(y^{\Delta}(t)\right)\right)^{\nabla}=-a(t)f(y(t)), t\in(0,T)_{\mathbb{T}^{\kappa}\cap\mathbb{T}_{\kappa}}, \phi_p\left(y^{\Delta}(0)\right)=0, y(T)=\psi(y):=\sum_{i=1}^{n}c_iy(\xi_i)$, where $\mathbb{T}$ is a time scale, $a:[0,T]_{\mathbb{T}}\to[0,+\infty)$ and $f:[0,+\infty)\to[0,+\infty)$ are continuous functions, and $\psi:\mathcal{C}_{\mathrm{ld}}([0,T]_{\mathbb{T}})\to\mathbb{R}$ is a given functional. We show that even if some of the c_i's are negative, the boundary value problem may still admit a positive solution. Our results extend and generalize some recent results on this type of problem, and we illustrate this by way of a numerical example.

1. Introduction

In this work we consider a second-order delta–nabla dynamic equation of the form

$$\left(\phi_p(y^{\Delta}(t))\right)^{\nabla}=-a(t)f(y(t)),\quad t\in(0,T)_{\mathbb{T}^{\kappa}\cap\mathbb{T}_{\kappa}}\tag{1.1}$$

subject to the boundary conditions

$$\phi_p(y^{\Delta}(0))=0,\tag{1.2}$$

$$y(T)=\psi(y):=\sum_{i=1}^{n}c_iy(\xi_i),\tag{1.3}$$

where $\mathbb{T}$ is a time scale, $a:[0,T]_{\mathbb{T}}\to[0,+\infty)$ and $f:[0,+\infty)\to[0,+\infty)$ are continuous functions, and $\psi:\mathcal{C}_{\mathrm{ld}}([0,T]_{\mathbb{T}})\to\mathbb{R}$ is a given functional. As indicated by (1.3) above, in our treatment, $\psi(y)$ will always have the multipoint form

$$\psi(y):=\sum_{i=1}^{n}c_iy(\xi_i),\tag{1.4}$$

for some finite sequence $\{\xi_i\}_{i=1}^{n}\subseteq(0,T)_{\mathbb{T}}$. It is certainly possible to take ψ to be a more general nonlocal condition than studied here, but the study of the case in which ψ is a multipoint condition should suffice to illustrate the main point of this work. We should also note that throughout this work we adopt the standard notation $[a,b]_{\mathbb{T}}:=[a,b]\cap\mathbb{T}$, and we also assume throughout that $0\in\mathbb{T}_{\kappa}, T\in\mathbb{T}^{\kappa}$, and $\mathbb{T}$ is a closed subset of $\mathbb{R}$, of course.

The function $\phi_p(\cdot)$ appearing in (1.1) is the one-dimensional p-Laplacian, which is defined by $\phi_p(z):=|z|^{p-2}z$, for $p>1$. Importantly, ϕ_p is multiplicative, for $\phi_p(z_1z_2)=\phi_p(z_1)\phi_p(z_2)$, for each z_1 and z_2 for which ϕ_p is defined; this is easy to

* Tel.: +1 402 472 3731.
E-mail address: s-cgoodri4@math.unl.edu.

0893-9659/$ – see front matter
doi:10.1016/j.aml.2011.08.005

check. Also of significant importance in the sequel is the elementary fact that $\phi_p^{-1}(z) = \phi_q(z)$, where p and q are Hölder conjugate—that is, $\frac{1}{p} + \frac{1}{q} = 1$.

To contextualize appropriately problem (1.1)–(1.3), we remark first that both continuous and discrete p-Laplacians arise in a number of natural settings in applied mathematics. For example, they occur in elasticity theory, fluid mechanics, and reaction–diffusion problems—see [1,2] and the references therein. From a more mathematical perspective, boundary value problems involving the p-Laplacian have been extensively studied over the past couple of decades, particularly in terms of the existence of one or more positive solutions—for example, see [3–15] and the references therein. In particular, we mention that second-order p-Laplacian BVPs on a time scale apparently were first studied in a work by Anderson et al. [16]. Secondly, the study of dynamic equations on time scales has rapidly developed since 1990 when Stefan Hilger published his seminal paper on time scales [17]. In the sequel, we assume a basic familiarity with time scales, and we refer the reader to the monographs on the subject by Bohner and Peterson [18,19].

The main contribution of this work is to use some recent ideas of Infante and Webb [20] to show that problem (1.1)–(1.3) may have at least one positive solution even if certain of the c_i coefficients appearing in (1.3) are *negative*. In particular, this will extend and generalize some recent work—specifically, He [21], Liu and Ge [7], Sang and Su [22], Su and Li [23], Sun and Li [24], and Zhu and Zhu [25].

2. Preliminaries

In this section we wish to fix our framework for problem (1.1)–(1.3). In particular, our strategy is to write a representation for a solution to problem (1.1)–(1.3) in terms of the fixed point of an appropriate operator. We will then show that this operator satisfies certain properties when acting on an appropriate cone. We introduce first the assumption that we shall make about $a(t)$ and $f(y)$.

F1: The continuous function $a(t)$ is not zero on $\left[\frac{T}{4}, \frac{3T}{4}\right]_{\mathbb{T}}$, whereas the continuous function $f(y)$ satisfies $K_1 \le f(y) \le K_2$, where $0 < K_1 < K_2$ are constants, for all $y \in [0, +\infty)$—that is, f is a bounded, positive function for all $y \ge 0$.

Remark 2.1. It should be noted that our condition (F1) is somewhat different than in other papers on problem (1.1)–(1.3). Largely this stems from our technique in the sequel insofar as overcoming the difficulty that negative c_i coefficients pose.

We also shall need to add some appropriate restrictions on the nonlocal term $\psi(y)$. We state these below.

G1: We find that $1 > \sum_{i=1}^{n} |c_i| \ge 0$.
G2: Define the sets $\mathscr{P}$ and $\mathscr{N}$ by $\mathscr{P} := \{i : c_i > 0\}$ and $\mathscr{N} := \{i : c_i < 0\}$. Then we find that

$$\phi_q(K_1)\sum_{i\in\mathscr{P}} c_i \int_{\xi_i}^{T} \phi_q\left(\int_0^s a(\tau)\nabla\tau\right)\Delta s + \phi_q(K_2)\sum_{i\in\mathscr{N}} c_i \int_{\xi_i}^{T} \phi_q\left(\int_0^s a(\tau)\nabla\tau\right)\Delta s \ge 0, \tag{2.1}$$

where K_1 and K_2 are the constants from condition (F1).

Let $\mathscr{B}$ be the Banach space $\mathcal{C}_{\mathrm{ld}}([0, T]_{\mathbb{T}}, \mathbb{R})$ when equipped with the usual supremum norm, $\|\cdot\|$. We are interested in the operator $S : \mathscr{B} \to \mathscr{B}$, where S is defined by

$$(Sy)(t) := \int_t^T \phi_q\left(\int_0^s a(\tau)f(y(\tau))\nabla\tau\right)\Delta s + \psi(y). \tag{2.2}$$

We see, for instance, that

$$(Sy)^{\Delta}(t) = -\phi_q\left(\int_0^t a(\tau)f(y(\tau))\nabla\tau\right) \tag{2.3}$$

so

$$\left(\phi_p((Sy)^{\Delta}(t))\right)^{\nabla} = -a(t)f(y(t)); \tag{2.4}$$

moreover, $\phi_p((Sy)^{\Delta}(0)) = 0$ and $(Sy)(T) = \psi(y)$. Therefore, a fixed point of (2.2) will satisfy the boundary value problem (1.1)–(1.3).

Remark 2.2. It is certainly possible to solve for each $y(\xi_i)$ in terms of integral expressions involving a and f. This allows us to rewrite (2.2) in terms of a and f only. Many works on problem (1.1) do just that—see, for example, [22,23]. However, for our purposes in the sequel, we shall not need this. Therefore, we work only with the form of S given in (2.2).

We now prove a lemma.

Lemma 2.3. *Let S be the operator defined in (2.2). Assume both that $y \ge 0$ and that $\psi(y) \ge 0$. Then it follows that there is a constant $\gamma \in (0, 1)$ such that*

$$\min_{t\in\left[\frac{T}{4}, \frac{3T}{4}\right]_{\mathbb{T}}} (Sy)(t) \ge \gamma\|Sy\|. \tag{2.5}$$

Proof. The function $(af)(t, y)$ is nonnegative for all t and y, $\psi(y) \geq 0$ by assumption, and from (2.3) we have that $(Sy)^{\Delta}(t) \leq 0$. Hence, it follows that $\|Sy\| = (Sy)(0)$, for any y, and that $\min_{t\in\left[\frac{T}{4},\frac{3T}{4}\right]_{\mathbb{T}}}(Sy)(t) = (Sy)(t_2)$, where we have put $t_2 := \max\left\{t \in \mathbb{T} : t \in \left[\frac{T}{4}, \frac{3T}{4}\right]_{\mathbb{T}}\right\}$. In particular, since f is bounded away from zero we know that each of these quantities is strictly greater than zero. Therefore, there must exist a number $\gamma \in (0, 1)$ such that (2.5) holds. Note that γ is strictly less than unity since $(Sy)(t)$ is strictly decreasing on $\left[\frac{T}{4}, \frac{3T}{4}\right]_{\mathbb{T}}$. □

Recalling the Banach space $\mathcal{B}$ from before, let us now introduce the cone that we shall use in the sequel. It is

$$\mathcal{K} := \left\{y \in \mathcal{B} : y \geq 0,\ \min_{t\in\left[\frac{T}{4},\frac{3T}{4}\right]_{\mathbb{T}}} y(t) \geq \gamma\|y\|,\ \psi(y) \geq 0\right\}, \tag{2.6}$$

which is evidently inspired by the recent work of Infante and Webb [20]. We now prove a preliminary result regarding the operator S. This will be essential in the sequel.

Lemma 2.4. *Let S be the operator defined in (2.2). If ψ satisfies conditions* (G1)–(G2)*, then $S : \mathcal{K} \to \mathcal{K}$.*

Proof. The fact that $\psi(Sy) \geq 0$ whenever $y \in \mathcal{K}$ follows from the observation that

$$\begin{aligned}
\psi(Sy) &= \sum_{i=1}^{n} c_i(Sy)(\xi_i) = \sum_{i=1}^{n} c_i\left[\int_{\xi_i}^{T}\phi_q\left(\int_0^s a(\tau)f(y(\tau))\nabla\tau\right)\Delta s + \psi(y)\right]\\
&= \psi(y)\sum_{i=1}^{n} c_i + \sum_{i=1}^{n} c_i\int_{\xi_i}^{T}\phi_q\left(\int_0^s a(\tau)f(y(\tau))\nabla\tau\right)\Delta s\\
&\geq \psi(y)\sum_{i=1}^{n} c_i + \sum_{i\in\mathscr{P}} c_i\int_{\xi_i}^{T}\phi_q\left(\int_0^s a(\tau)K_1\nabla\tau\right)\Delta s + \sum_{i\in\mathscr{N}} c_i\int_{\xi_i}^{T}\phi_q\left(\int_0^s a(\tau)K_2\nabla\tau\right)\Delta s\\
&= \psi(y)\sum_{i=1}^{n} c_i + \phi_q(K_1)\sum_{i\in\mathscr{P}} c_i\int_{\xi_i}^{T}\phi_q\left(\int_0^s a(\tau)\nabla\tau\right)\Delta s + \phi_q(K_2)\sum_{i\in\mathscr{N}} c_i\int_{\xi_i}^{T}\phi_q\left(\int_0^s a(\tau)\nabla\tau\right)\Delta s\\
&\geq 0,
\end{aligned} \tag{2.7}$$

where $\mathscr{P}$ and $\mathscr{N}$ retain their meanings from earlier. Note that the final inequality in (2.7) follows from assumptions (G1)–(G2) as well as the fact that $\psi(y) \geq 0$ whenever $y \in \mathcal{K}$. On the other hand, by the form of S together with the fact that a and f are nonnegative, it is clear that $(Sy)(t) \geq 0$ whenever $y \in \mathcal{K}$ since $\psi(y) \geq 0$ whenever $y \in \mathcal{K}$. Finally, the result of Lemma 2.3 shows that whenever $y \in \mathcal{K}$, we find that

$$\min_{t\in\left[\frac{T}{4},\frac{3T}{4}\right]_{\mathbb{T}}}(Sy)(t) \geq \gamma\|Sy\|.$$

Therefore, we conclude that $S : \mathcal{K} \to \mathcal{K}$, which completes the proof. □

We now state without proof a fact that will be useful in Section 3; its proof is straightforward.

Lemma 2.5. *Let $f(y)$ satisfy condition* (F1)*. Then* $\lim_{y\to 0^+}\frac{f(y)}{\phi_p(y)} = +\infty$.

Finally, we end this section by recalling as a preliminary lemma the Krasnosel'skiĭ fixed point theorem (see [26]), which we shall use to prove our existence result in Section 3.

Lemma 2.6. *Let $\mathcal{B}$ be a Banach space and let $\mathcal{K} \subseteq \mathcal{B}$ be a cone. Assume that Ω_1 and Ω_2 are bounded open sets contained in $\mathcal{B}$ such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subseteq \Omega_2$. Assume, further, that $T : \mathcal{K} \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \to \mathcal{K}$ is a completely continuous operator. If either*

1. *$\|Ty\| \leq \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_1$ and $\|Ty\| \geq \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_2$; or*
2. *$\|Ty\| \geq \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_1$ and $\|Ty\| \leq \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_2$;*

then T has at least one fixed point in $\mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. The existence of a positive solution

We now state and prove an existence result for problem (1.1)–(1.3).

Theorem 3.1. *Suppose that conditions* (F1) *and* (G1)–(G2) *hold. Then problem* (1.1)–(1.3) *has at least one positive solution.*

Proof. Lemma 2.4 shows that $S : \mathcal{K} \to \mathcal{K}$. In addition, a standard argument involving the Arzela–Ascoli theorem, which we omit, shows that S is a completely continuous operator. Now, pick a number $c > 0$ such that

$$c \int_{\left[\frac{T}{4}, \frac{3T}{4}\right]_{\mathbb{T}}} \gamma \phi_q \left(\int_0^s a(\tau) \nabla \tau \right) \Delta s \geq 1. \tag{3.1}$$

Lemma 2.5 shows that there is $r_1 > 0$ such that whenever $0 < y \leq r_1$, we find that $f(y) \geq \phi_p(y)\phi_p(c)$. Put $\Omega_1 := \{y \in \mathcal{B} : \|y\| < r_1\}$. Then for $y \in \mathcal{K} \cap \partial\Omega_1$,

$$\begin{aligned} \|Sy\| &= (Sy)(0) = \int_0^T \phi_q \left(\int_0^s a(\tau) f(y(\tau)) \nabla \tau \right) \Delta s + \psi(y) \\ &\geq \int_0^T \phi_q \left(\int_0^s a(\tau)\phi_p(c)\phi_p(y(\tau)) \nabla \tau \right) \Delta s \\ &\geq c\|y\| \int_{\left[\frac{T}{4}, \frac{3T}{4}\right]_{\mathbb{T}}} \gamma \phi_q \left(\int_0^s a(\tau) \nabla \tau \right) \Delta s \geq \|y\|, \end{aligned} \tag{3.2}$$

where we have used the fact that $\psi(y) \geq 0$ since $y \in \mathcal{K}$ to obtain the first inequality, and where we have used the choice of c as given in (3.1) to obtain the last inequality. Thus, (3.2) implies that $\|Sy\| \geq \|y\|$ whenever $y \in \mathcal{K} \cap \partial\Omega_1$.

Conversely, by condition (G1) we can choose $\varepsilon > 0$ such that

$$0 \leq \sum_{i=1}^{n} |c_i| < 1 - \varepsilon. \tag{3.3}$$

By assumption (F1), we know that $f(y)$ is bounded for all $y \geq 0$. Therefore, there is $r_2^* > 0$ sufficiently large such that $f(y) \leq \phi_p(r_2^*)$, for all $y \geq 0$. Put

$$r_2 := \max \left\{ \frac{2}{\varepsilon} r_1, \frac{r_2^*}{\varepsilon} \int_0^T \phi_q \left(\int_0^s a(\tau) \nabla \tau \right) \Delta s \right\}$$

and then define $\Omega_2 := \{y \in \mathcal{B} : \|y\| < r_2\}$; of course, we have that $r_2 > r_1$ since $0 < \varepsilon < 1$. It follows that whenever $y \in \mathcal{K} \cap \partial\Omega_2$, we have

$$\begin{aligned} \|Sy\| &= (Sy)(0) \leq \sum_{i=1}^{n} c_i y(\xi_i) + \int_0^T \phi_q \left(\int_0^s \phi_p(r_2^*) a(\tau) \nabla \tau \right) \Delta s \\ &\leq r_2 \sum_{i=1}^{n} |c_i| + r_2\varepsilon \leq r_2(1-\varepsilon) + r_2\varepsilon = r_2, \end{aligned} \tag{3.4}$$

whence $\|Sy\| \leq \|y\|$. Consequently, from (3.2) and (3.4) we conclude from Lemma 2.6 that S has a fixed point in $\mathcal{K} \cap (\overline{\Omega_2} \setminus \Omega_1)$. As this fixed point is a solution of problem (1.1)–(1.3), the proof is finished. $\square$

4. A numerical example

We now present a numerical example to illustrate the use of Theorem 3.1.

Example 4.1. Consider the problem

$$(|y^{\Delta}(t)|^3 y^{\Delta}(t))^{\nabla} = -(t+1)^2(3e^{-y(t)} + 5\sin y(t) + 10), \quad t \in [1, 19]_{\mathbb{N}_0} \tag{4.1}$$

subject to the boundary conditions

$$\phi_p(y^{\Delta}(0)) = 0, y(20) = \frac{3}{10} y(3) - \frac{1}{6} y(14) + \frac{1}{20} y(19). \tag{4.2}$$

Obviously here we have made the following declarations.

$$\begin{aligned} &f(y) = 3e^{-y} + 5\sin y + 10 \\ &a(t) = (t+1)^2 \\ &\psi(y) = \frac{3}{10} y(3) - \frac{1}{6} y(14) + \frac{1}{20} y(19) \\ &p = 5 \\ &\mathbb{T} = \mathbb{Z} \\ &T = 20 \end{aligned} \tag{4.3}$$

It is obvious that condition (F1) holds with $K_1 := 4$ and $K_2 := 18$, say. Condition (G1) evidently holds since $1 > \sum_{i=1}^{n} |c_i| = \frac{31}{60} \geq 0$. Finally, condition (G2) can be checked numerically to deduce that

$$\phi_{\frac{5}{4}}(4) \sum_{i \in \mathscr{P}} c_i \int_{\xi_i}^{T} \phi_{\frac{5}{4}} \left(\int_0^s a(\tau) \nabla \tau \right) \Delta s + \phi_{\frac{5}{4}}(18) \sum_{i \in \mathscr{N}} c_i \int_{\xi_i}^{T} \phi_{\frac{5}{4}} \left(\int_0^s a(\tau) \nabla \tau \right) \Delta s \approx 67.232 \geq 0. \tag{4.4}$$

Therefore, we conclude from Theorem 3.1 that problem (4.1)–(4.2) has at least one positive solution.

Remark 4.2. Note, importantly, that not all of the coefficients in the multipoint term in (4.2) are nonnegative. Therefore, this example could not be handled by using the results given recently in [7,21–25] among others. Thus, our results extend and generalize the results presented in those papers.

Remark 4.3. Example 4.1 remains true on the time scale $\mathbb{T} = \mathbb{R}$, as an easy numerical calculation reveals.

5. Conclusions

In this work we have shown how some of the recently introduced ideas of Infante and Webb [20] can be used in the time scales setting to deduce the existence of at least one positive solution to a second-order delta–nabla BVP involving the one-dimensional p-Laplacian. By means of a numerical example we have shown how our results explicitly extend other recent results on this type of problem.

Finally, by way of a conclusion, we suggest three avenues for additional study of this type of problem. Firstly, it would be interesting to see in what ways condition (F1) can be modified and weakened even in the context of negative coefficients in the multipoint condition. It may also be interesting to attempt to extend problem (1.1)–(1.3) to the fractional setting since the fractional calculus is an area that has recently been gaining considerable traction even in the discrete and, more generally, time scales settings (see, for example, [27–33]). Thirdly, it might be interesting to extend these results to the more general setting of an increasing homeomorphism and homomorphism—see, for example, [10–12]. In any case, any of these avenues might produce interesting future results for this problem.

References

[1] J. Diaz, F. de Thelin, On a nonlinear parabolic problem arising in some models related to turbulent flows, SIAM J. Math. Anal. 25 (1994) 1085–1111.
[2] P.Y.H. Pang, Y. Wang, J. Yin, Periodic solutions for a class of reaction–diffusion equations with p-Laplacian, Nonlinear Anal. RWA 11 (2010) 323–331.
[3] D.R. Anderson, Existence of solutions for a first-order p-Laplacian BVP on time scales, Nonlinear Anal. 69 (2008) 4521–4525.
[4] R.I. Avery, J. Henderson, Existence of three positive pseudo-symmetric solutions for a one dimensional discrete p-Laplacian, J. Difference Equ. Appl. 10 (2004) 529–539.
[5] P. Candito, N. Giovannelli, Multiple solutions for a discrete boundary value problem involving the p-Laplacian, Comput. Math. Appl. 56 (2008) 959–964.
[6] F. Geng, D. Zhu, Multiple results of p-Laplacian dynamic equations on time scales, Appl. Math. Comput. 193 (2007) 311–320.
[7] Y. Liu, W. Ge, Twin positive solutions of boundary value problems for finite difference equations with p-Laplacian operator, J. Math. Anal. Appl. 278 (2003) 551–561.
[8] D. Wang, W. Guan, Three positive solutions of boundary value problems for p-Laplacian difference equations, Comput. Math. Appl. 55 (2008) 1943–1949.
[9] Y. Wang, C. Hou, Existence of multiple positive solutions for one-dimensional p-Laplacian, J. Math. Anal. Appl. 315 (2006) 114–153.
[10] W. Han, Z. Jin, S. Kang, Existence of positive solutions of nonlinear m-point BVP for an increasing homeomorphism and positive homomorphism on time scales, J. Comput. Appl. Math. 223 (2009) 188–196.
[11] W. Han, S. Kang, Several existence theorems of multiple positive solutions of nonlinear m-point BVP for an increasing homeomorphism and homomorphism on time scales, Bound. Value Probl. (2009) Art. ID 584145, 19 pp.
[12] Y. Sang, H. Su, F. Xu, Positive solutions of nonlinear m-point BVP for an increasing homeomorphism and homomorphism with sign changing nonlinearity on time scales, Comput. Math. Appl. 58 (2009) 216–226.
[13] C.S. Goodrich, Existence of a positive solution to a first-order p-Laplacian BVP on a time scale, Nonlinear Anal. 74 (2011) 1926–1936.
[14] C. Song, X. Gao, Positive solutions for third-order p-Laplacian functional dynamic equations on time scales, Bound. Value Probl. (2011) Art. ID 585378, 11 pp.
[15] Y. Li, T. Zhang, Multiple positive solutions for second-order p-Laplacian dynamic equations with integral boundary conditions, Bound. Value Probl. (2011) Art. ID 867615, 17 pp.
[16] D.R. Anderson, R.I. Avery, J. Henderson, Existence of solutions for a one-dimensional p-Laplacian on time scales, J. Difference Equ. Appl. 10 (2004) 889–896.
[17] S. Hilger, Analysis on measure chains—a unified approach to continuous and discrete calculus, Results Math. 18 (1990) 18–56.
[18] M. Bohner, A.C. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhäuser, Boston, 2001.
[19] M. Bohner, A.C. Peterson (Eds.), Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston, 2003.
[20] G. Infante, J.R.L. Webb, Positive solutions of nonlocal boundary value problems: a unified approach, J. Lond. Math. Soc. 74 (2) (2006) 673–693.
[21] Z. He, Double positive solutions of three-point boundary value problems for p-Laplacian dynamic equations on time scales, J. Comput. Appl. Math. 182 (2005) 304–315.
[22] Y. Sang, H. Su, Several existence theorems of nonlinear m-point boundary value problem for p-Laplacian dynamic equations on time scales, J. Math. Anal. Appl. 340 (2008) 1012–1026.
[23] Y.H. Su, W.T. Li, Triple positive solutions of m-point BVPs for p-Laplacian dynamic equations on time scales, Nonlinear Anal. 69 (2008) 3811–3820.
[24] H.R. Sun, W.T. Li, Existence theory for positive solutions to one-dimensional p-Laplacian boundary value problems on time scales, J. Differential Equations 240 (2007) 217–248.
[25] Y. Zhu, J. Zhu, The multiple positive solutions for p-Laplacian multipoint BVP with sign changing nonlinearity on time scales, J. Math. Anal. Appl. 344 (2008) 616–626.
[26] R. Agarwal, M. Meehan, D. O'Regan, Fixed Point Theory and Applications, Cambridge University Press, Cambridge, 2001.
[27] F.M. Atici, P.W. Eloe, Initial value problems in discrete fractional calculus, Proc. Amer. Math. Soc. 137 (2009) 981–989.

[28] F.M. Atici, P.W. Eloe, Two-point boundary value problems for finite fractional difference equations, J. Difference Equ. Appl. 17 (2011) 445–456.
[29] Z. Bai, H. Lü, Positive solutions for boundary value problem of nonlinear fractional differential equation, J. Math. Anal. Appl. 311 (2005) 495–505.
[30] N.R.O. Bastos, et al., Discrete-time fractional variational problems, Signal Process. 91 (2010) 513–524.
[31] C.S. Goodrich, Continuity of solutions to discrete fractional initial value problems, Comput. Math. Appl. 59 (2010) 3489–3499.
[32] C.S. Goodrich, Existence of a positive solution to a class of fractional difference equations, Appl. Math. Lett. 23 (2010) 1050–1055.
[33] C.S. Goodrich, Solutions to a discrete right-focal fractional boundary value problem, Int. J. Difference Equ. 5 (2010) 195–216.